JOURNAL OF APPROXIMATION THEORY **20**, 173–175 (1977)

Rational Approximation to e^x with Negative Zeros and Poles

D. J. NEWMAN*

Belfer Graduate School of Science, Yeshiva University, New York, New York 10033

Communicated by Oved Shisha

Received October 6, 1975

The problem of approximating e^x has long been a popular one. Assessing the order of approximation by *polynomials* was already considered by Bernstein, and of late many authors have treated the corresponding question for *rational functions*.

Just recently Saff and Reddy asked, respectively, about polynomial and rational function approximation to e^x when the zeros and poles are restricted to the negative axis. These are the questions we treat in this paper. We determine the exact order for the case of polynomials while we only obtain rather crude bounds for the rational case. These bounds, however, do establish the qualitative facts that (A) rational approximation is *far better* than polynomial approximation, and (B) the restriction on the zeros and poles does make the approximation *far worse*.

For convenience we work with the interval [0, 1] although any other finite interval could have been used. Our precise statements are

THEOREM 1. (I) $|e^x - (1 + (x/n))^n| \leqslant 2/n$ *on* [0, 1].

(II) *If* $\deg p(x) \leqslant n$ *and* $p(x)$ *has all real zeros then at one of the three points* 0, $\frac{1}{2}$, *or* 1 *we must have* $|e^x - p(x)| > 1/17n$.

THEOREM 2. (I). *There is a rational function*, $R(x)$, *of total degree* n *having negative* integers *for zeros and poles and such that, throughout* [0, 1], $|e^x - R(x)| \leqslant n^{-c \log n}$ (*c a fixed positive constant*).

(II) *There is no rational function*, $r(x)$, *of total degree* n *having negative zeros and poles and such that, throughout* [0, 1], $|e^x - r(x)| \leqslant 512^{-n}$.

As we predicted above Theorem 2(I) compared to Theorem 1(II) shows (A), the superiority of rational over polynomial approximation, while Theorem 2(II) compared to the fact that $\sum_{k=0}^{n} (x^k/k!)$ lies within $1/n!$ of e^x shows (B), that the negativity condition does indeed worsen the approximation.

* Supported by NSF (MPS) 75-08002.

ISSN 0021-9045

Proof of Theorem 1(I). This is a standard exercise. We have

$$\log\left(1+\frac{x}{n}\right)=\frac{x}{n}-\frac{x^2}{2n^2}+\frac{x^3}{3n^3}-\cdots$$

so that

$$\log\frac{(1+(x/n))^n}{e^x}=-\frac{x^2}{2n}+\frac{x^3}{3n^2}-\cdots\leqslant\frac{1}{2n}$$

by the alternating series theorem. Hence, indeed

$$|(1+(x/n))^n-e^x|\leqslant e\left|\frac{(1+(x/n))^n}{e^x}-1\right|\leqslant e(e^{1/2n}-1)<2/n.$$

Proof of Theorem 1(II).

LEMMA. *Suppose* deg $p(x)\leqslant n$, $p(x)$ *has all real zeros, and* $p(x)>0$ *on* $[a, b]$, *then* $(p(x))^{1/n}$ *is concave on* $[a, b]$.

Proof. We write $p(x)=c\prod_\alpha(x+\alpha)$ and differentiate twice, obtaining

$$(p^{1/n})''=\frac{p^{1/n}}{n^2}\left(\left(\sum_\alpha\frac{1}{x+\alpha}\right)^2-n\sum_\alpha\frac{1}{(x+\alpha)^2}\right).$$

But $(\sum(1/(x+\alpha)))^2\leqslant n\sum_\alpha(1/(x+\alpha)^2)$ by Schwarz' inequality so that indeed $(p^{1/n})''\leqslant 0$.

We may assume that $n>1$, and we suppose that at $0, \frac{1}{2}, 1$, $|e^x-p(x)|\leqslant\epsilon$ so that $p(0)\geqslant 1-\epsilon$, $p(\frac{1}{2})\leqslant e^{1/2}+\epsilon<e^{1/2}/(1-\epsilon)$, $p(1)\geqslant e-\epsilon>e(1-\epsilon)$. Hence $(p(0))^{1/n}-2(p(\frac{1}{2}))^{1/n}+(p(1))^{1/n}>e^{1/2n}(1-\epsilon)^{1/n}[e^{1/2n}+e^{-1/2n}-2/(1-\epsilon)^{2/n}]$ and the left-hand side is nonpositive, by our lemma. Hence $1/(1-\epsilon)^{2/n}>\cosh(1/2n)>1+(1/8n^2)$, and so $1/(1-\epsilon)>(1+(1/8n^2))^{n/2}\geqslant 1+(1/16n)$ which gives $\epsilon>1/(16n+1)>1/17n$ as required.

For both halves of Theorem 2 we use the formula,

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{1}{k+s}=\frac{m!}{s(s+1)\cdots(s+m)}\quad F$$

which is proved, e.g., by partial, fractions (or via the B-function).

Proof of Theorem 2(I). In F write N/t for s, t in $[0, 1]$ and obtain

$$1-\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}\frac{N}{k}\frac{1}{t+(N/k)}$$
$$=\frac{m!\,t^mN}{(N+t)(N+2t)\cdots(N+mt)}\leqslant\frac{m!}{N^m}$$

so that integration from 0 to x in [0, 1] yields

$$x - \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} \frac{N}{k} \log\left(1 + \frac{k}{N} x\right) \leqslant \frac{m!}{N^m}.$$

We now choose $N =$ l.c.m. $\{1, 2, 3, \ldots, m\}$ and conclude that the rational function

$$R(x) = \prod_{k=1}^{m} \left(1 + \frac{k}{N} x\right)^{(-1)^{k-1}\binom{m}{k}\frac{N}{k}}$$

approximates e^x to within $Am!/N^m$. From prime number theory it is known, for $m > 1$, that $e^{\alpha m} < N < e^{\beta m}$, α, β fixed positive constants. Hence, since $\deg R(x) \leqslant 2^m N$ this degree can be kept $\leqslant n$ while choosing $m > \gamma \log n$. This done allows the error estimate : $Am!/N^m < e^{-\delta m^2} < e^{-c(\log n)^2}$ as required.

Proof of Theorem 2(II). Here we use formula F in quite a different way. Set in $s = m + mt$, integrate to obtain

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \log\left(1 + \frac{mu}{m+k}\right)$$
$$= \frac{1}{\binom{2m}{m}} \int_0^u \frac{dt}{(1+t)(1+(mt/(m+1)))\cdots(1+(mt/(m+m)))},$$

and note that, for $u > 0$, this right side is bounded by

$$\frac{1}{\binom{2m}{m}} \int_0^\infty \frac{dt}{(1+(t/2))^{m+1}} = \frac{2}{m\binom{2m}{m}}.$$

Also, using formula F with $s = m$ gives $\sum_0^m (-1)^k \binom{m}{k}(m/(k+m)) = 1/\binom{2m}{m}$.

Now suppose that we did have a rational function $r(x)$, $r(x) = e^{-c} \prod_{i=1}^{n} (1 + xu_i)^{\epsilon_i}$, $\epsilon_i = \pm 1$, $u_i \geqslant 0$, such that $|e^x - r(x)| < \epsilon$. Thus we would have $c - \sum_{i=1}^{n} \epsilon_i \log(1 + xu_i) + x \leqslant \epsilon$ and, in particular, $c - \sum_{i=1}^{n} \epsilon_i \log(1 + (m/(m+k))\, u_i) + (m/(m+k)) \leqslant \epsilon$ for $k = 0, 1, \ldots, m$. Now apply Δ^m to both sides. By our two previous estimates we obtain

$$-n \cdot \frac{2}{m\binom{2m}{m}} + \frac{1}{\binom{2m}{m}} \leqslant 2^m \epsilon.$$

Finally, choosing $m = 3n$ yields

$$2^{3n}\epsilon \geqslant \frac{1}{3\binom{6n}{3n}} > \frac{1}{2^{6n}} \qquad \text{or} \qquad \epsilon > 512^{-n}.$$

Q.E.D.